Computability of Diagrammatic Theories for Normative Positions

Matteo PASCUCCI [a], Giovanni SILENO [b,1]
[a] *Slovak Academy of Sciences, Bratislava, Slovakia*
[b] *University of Amsterdam, Amsterdam, The Netherlands*

Abstract. Normative positions are sometimes illustrated in diagrams, in particular in didactic contexts. Traditional examples are the Aristotelian polygons of opposition for deontic modalities (squares, triangles, hexagons, etc.), and the Hohfeldian squares for obligative and potestative concepts. Relying on previous work, we show that Hohfeld's framework can be used as a basis for developing several Aristotelian polygons and more complex diagrams. Then, we illustrate how logical theories of increasing strength can be built based on these diagrams, and how those theories enable us to determine in a computably efficient way whether a set of normative positions can be derived from another set of normative positions.

Keywords. Computable Normative Theories; Diagrams; Hohfeldian Relationships; Normative Positions; Polygons of Opposition

Introduction

Diagrams are acknowledged to be effective instruments for didactic purposes: they are commonly used to illustrate in an intuitive and accessible way the most various conceptual models, procedures, systems. Examples abound in engineering and theoretical sciences, but archetypes of diagrams include also the *Aristotelian square of opposition* [1], used in philosophical, linguistic, literary, and semiotic studies; and, within legal studies, the *two squares of fundamental normative relationships* due to Hohfeld [2,3]. Reference to regular shapes may be not by chance; cognitive studies show that symmetries facilitate perception of structure, memorization and recall [4]. Indeed, the motivating intuition behind this research is that diagrams may be useful to create user-friendly interfaces for the analysis of legal/contractual constructs. Rather than inspecting hundreds of sentences in the text of a contract, a subject may more easily figure out her normative relations (duties, rights, etc.) with the other parties by navigating or exploring a diagram-construed model. One may also ask whether these diagrams have similarly interesting computational properties; if this is the case, unveiling diagrammatic theories may benefit on multiple levels. For this reason, the present work focuses on the computational treatment of *Aristotelian diagrams* to represent normative positions, building upon previous interpretations of Hohfeld's squares relating these two types of diagrams [5,6,7,8,9].

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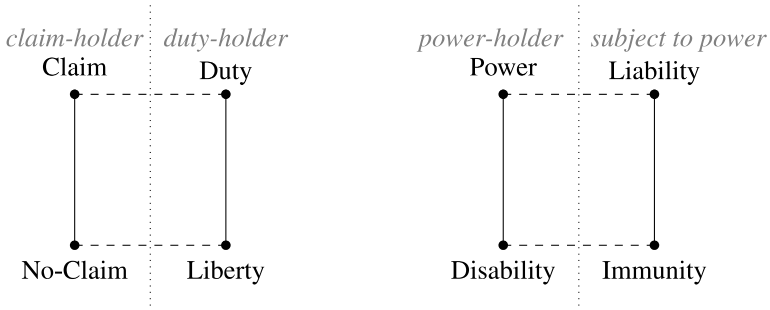

Figure 1. The two Hohfeldian squares: (left) the deontic square, (also obligative or of the first-order), and (right) the potestative square (or of the second-order).

The paper proceeds as follows. Section 1 introduces the notion of a normative position, the two Hohfeldian squares, and the notion of an Aristotelian diagram. Section 2 provides a formal language to encode normative positions and, reorganizing and extending previous results in [8], analyses the representation of Hohfeld's deontic square and of three versions of Hohfeld's potestative square (change-centered, force-centered and outcome-centered) in terms of Aristotelian diagrams. Section 3 shows how to develop logical theories over a diagram relying on the syntactic notion of an *inference tree*. These will be called *diagrammatic theories*. The analysis of a complex diagram which combines the three potestative squares of opposition is used as an example. Section 4 focuses on the computational dimension of diagrams, many aspects of which have recently gained attention in the literature (see, e.g., [10]). In particular, it shows that diagrammatic theories enable one to compute in polynomial time whether a finite set of normative positions can be derived from another.

1. Normative positions on diagrams

A *normative position* is described by a statement involving a normative concept, one or more normative parties related to that concept and a certain type of behaviour of one of those parties. For instance, consider the following:

- it is obligatory for the company to pay an annual fee;
- borrowers have a duty towards lenders to bring back the relevant goods;
- by accepting a sale offer, a buyer creates a duty upon the seller to deliver.

Two families of normative positions have been extensively analysed by Hohfeld [2,3]. He proposed to graphically visualize their relations via two diagrams, a *deontic* and a *potestative* square, which are reproduced in Fig. 1.

Many formal accounts of Hohfeld's analysis have been proposed (see, e.g., Lindahl [11], Makinson [12], and, more recently, Markovich [13] or Sileno and Pascucci [14]). Furthermore, some works [5,6,7] have shown that relationships on Hohfeldian squares can be used to construct Aristotelian polygons of opposition. The fundamental advantage of the latter over Hohfeld's squares is that they *unambiguously express logical relations* of a certain kind between normative statements. Aristotelian polygons can be combined to form more complex figures, whence we will generally speak of *Aristotelian diagrams*. At the basis of an Aristotelian diagram are four types of logical relations [10].

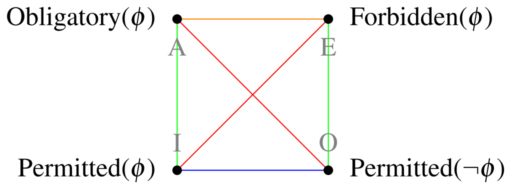

Figure 2. Aristotelian square for basic deontic modalities, with subaltern (green), contrary (orange), sub-contrary (blue) and contradictory (red) bindings, and named vertices (A, E, I, O). Sub--alternity is *directed*: from A to I, and from E to O, i.e., from orange towards blue lines.

Definition 1 (Aristotelian Relation)**.** *An* Aristotelian relation *between two sentences ϕ and ψ is either* subalternation, contrariety, sub-contrariety *or* contradiction, *where:*

- *ϕ is a subalternate of ψ iff the truth of ψ implies the truth of ϕ;*
- *ϕ and ψ are contrary iff at most one between ϕ and ψ can be true;*
- *ϕ and ψ are sub-contrary iff at most one between ϕ and ψ can be false;*
- *ϕ and ψ are contradictory iff ϕ is true precisely when ψ is false.*

Definition 2 (Aristotelian Diagram)**.** *Given a finite set of sentences Γ, an Aristotelian diagram over Γ is a graph whose vertices are labelled by elements of Γ. Each vertex v is labelled by a distinct sentence. An edge e of the graph connecting two vertices v_1 and v_2 is associated with an Aristotelian relation R_e between the sentences labelling v_1 and v_2.*

In the normative domain, the simplest example of Aristotelian diagram is the *square of opposition for basic deontic modalities* (Fig. 2).

2. Formalization

To enable a more rigorous analysis of normative positions, we here use a language $\mathscr{L}$ of first-order logic to encode sentences, following [11]. Moreover, refining and extending recent work [8], we show how to construct squares of oppositions starting from Hohfeld's squares and, subsequently, more complex Aristotelian diagrams.

2.1. Language

Language $\mathscr{L}$ has variables x, y etc. for normative parties and α, β, etc. for action types. It has constants p, q, etc. for normative parties and constants A, B, etc. for action types. Symbol $-$ (overline) denotes action complementation: $\overline{A}$ is the complement of A (the type of any action not instantiating A). We assume that $\overline{\overline{A}} = A$. Hohfeldian (and other) relations are encoded as n-ary predicates.[2] Finally, $\mathscr{L}$ has standard propositional connectives ($\neg$, $\wedge$, $\vee$, $\rightarrow$, $\equiv$) and quantifiers ($\forall$, $\exists$). We omit quantification over variables for normative parties, interpreting an expression $\phi(x,y,\ldots)$ as implicitly having the form $\forall x \forall y \ldots \phi(x,y,\ldots)$. Thus, while $\mathsf{Claim}(x,y,A)$ means "for all x, for all y: x has a claim that A be performed by y", $\mathsf{Claim}(p,q,A)$ means "p has a claim that A be performed by q".

[2]Sometimes the argument of a relation is a statement involving another relation. However, no quantification on such statements is employed; therefore, $\mathscr{L}$ remains a first-order language.

2.2. *First-order Hohfeldian relations*

The formal renderings of the fundamental deontic relations identified in Hohfeld's framework, for two normative parties p and q and an action type A, are the following: $\mathsf{Claim}(p,q,A)$, $\mathsf{Liberty}(p,q,A)$, $\mathsf{Duty}(p,q,A)$ and $\mathsf{NoClaim}(p,q,A)$. We can map all relationships to a single primitive, e.g. Claim:

$$\begin{aligned} \mathsf{NoClaim}(x,y,A) &\equiv \neg\mathsf{Claim}(x,y,A) \\ \mathsf{Duty}(y,x,A) &\equiv \mathsf{Claim}(x,y,A) \\ \mathsf{Liberty}(y,x,A) &\equiv \neg\mathsf{Claim}(x,y,\overline{A}) \end{aligned}$$

This choice leads to the following set of labels DR with respect to a given action type A:

$$\mathsf{DR} = \{\mathsf{Claim}(p,q,A), \mathsf{Claim}(p,q,\overline{A}), \neg\mathsf{Claim}(p,q,\overline{A}), \neg\mathsf{Claim}(p,q,A)\}$$

The set DR naturally gives rise to a deontic square of opposition. The only additional principle needed is the following, used to characterize *subalternate statements*: $\mathsf{Claim}(x,y,A) \rightarrow \neg\mathsf{Claim}(x,y,\overline{A})$, which can be seen as corresponding to the $\mathit{Obligatory}(\phi) \rightarrow \mathit{Permitted}(\phi)$ axiom used in deontic logics.

2.3. *Second-order Hohfeldian relations*

Potestative relations concern *actions* that trigger changes of first-order or even second-order relations, such as, for instance, an action B creating a duty for a party q with respect to a party p to perform an action A. A possible way of writing that p has such a power would be by means of a predicate expression $Ability(p,B,R)$ (cf. the predicate $has\_ability$ investigated in [14]), where R is a Hohfeldian relation issued at B's performance by p; for instance, $Ability(p,B,\mathsf{Claim}(p,q,A))$. Different characterizations of actions exist in human language, mapping to different levels of abstraction [15], e.g. the *behavioural* or *procedural* characterization, relating to the action task, or the *productive* characterization, relating to its outcome. In the following, we similarly provide different definitions of power constructed at different abstraction levels (force, outcome, change).

2.3.1. *Force-centered power*

At behavioural level, power relations can be seen in analogy to force fields determining attraction, repulsion, and absence of those (independence) at the occurrence of interventions ([7], [16, Ch.4]). To express this, we need to separate the *stimulus* component (a particular type of action, such as a verbal command) and the consequent target *manifestation* (e.g. a type of action that is due or expected on the basis of the stimulus, cf. the concept of *pliance*). If the action-manifestation is denoted by the action type symbol A, then, the action-stimulus can be conveniently represented via the symbol `"A"` to emphasize the shared connection between signal and expected performance.

If stimulus and manifestation converge, i.e. A is always performed in correspondence to its stimulus, we have a *positive force-centered* power:

$$\overset{\rightarrow\leftarrow}{\mathsf{Power}}(x,y,A) \equiv Ability(x, \texttt{"A"}, \mathsf{Claim}(x,y,A))$$

If stimulus and manifestation diverge, i.e. A is never performed in correspondence to its stimulus, we have a *negative force-centered* power:[3]

$$\overset{\leftarrow\rightarrow}{\mathsf{Power}}(x,y,A) \equiv Ability(x,\texttt{"A"},\mathsf{Claim}(x,y,\overline{A}))$$

From these concepts we can define a new set of potestative relations $\mathsf{PR}^{\leftrightarrows}$ as labels for a force-centered potestative square of opposition, obtained by taking into account all possible combinations of positive- vs. negative-force power and Boolean negation:

$$\mathsf{PR}^{\leftrightarrows} = \{\overset{\rightarrow\leftarrow}{\mathsf{Power}}(p,q,A), \overset{\leftarrow\rightarrow}{\mathsf{Power}}(p,q,A), \neg\overset{\leftarrow\rightarrow}{\mathsf{Power}}(p,q,A), \neg\overset{\rightarrow\leftarrow}{\mathsf{Power}}(p,q,A)\}$$

The subalternity is here captured by the logical principle: $\overset{\rightarrow\leftarrow}{\mathsf{Power}}(x,y,A) \rightarrow \neg\overset{\leftarrow\rightarrow}{\mathsf{Power}}(x,y,A)$ which is acceptable because otherwise the same stimulus `"A"` would generate two conflicting first-order relations.

2.3.2. *Outcome-centered power*

At the outcome or productive abstraction level, we may abstract the triggering action B, and focus only the output, e.g. $R = \mathsf{Claim}(p,q,A)$. *Positive outcome-centered* power, expressed in the form $\mathsf{Power}(p,q,A)$, means that p has the power of issuing a duty upon q to A. It can be defined via an existential quantification on the set of action types:

$$\mathsf{Power}(x,y,A) \equiv \exists\beta : Ability(x,\beta,\mathsf{Claim}(x,y,A))$$

We can similarly define a *negative outcome-centered* power (to release a duty to A) as:

$$\overline{\mathsf{Power}}(x,y,A) \equiv \exists\beta : Ability(x,\beta,\neg\mathsf{Claim}(x,y,A))$$

As before, we can form a set of potestative relations PR as labels for an outcome-centered potestative square of opposition:

$$\mathsf{PR} = \{\mathsf{Power}(p,q,A), \overline{\mathsf{Power}}(p,q,A), \neg\overline{\mathsf{Power}}(p,q,A), \neg\mathsf{Power}(p,q,A)\}$$

where subalternity is captured by: $\mathsf{Power}(p,q,A) \rightarrow \neg\overline{\mathsf{Power}}(p,q,A)$. This principle can be explained as such: to create a duty, this duty needs not to be holding: $\mathsf{Power}(x,y,A) \rightarrow \neg\mathsf{Claim}(x,y,A)$. Dually, to release a duty, this needs to exist: $\overline{\mathsf{Power}}(x,y,A) \rightarrow \mathsf{Claim}(x,y,A)$; its contrapositive provides the subalternity relation.

2.3.3. *Change-centered power*

Given a target relation R, e.g. a due performance $\mathsf{Claim}(p,q,A)$, one can define power also as the ability of p to *affect* q in any sense with respect to this relation R. This proposal, originally made by O'Reilly [6], can be reframed in our framework as the ability of changing q's position in the (e.g. deontic) square of opposition of which R is part. Using the proposed notation we have:

$$\begin{aligned}\mathsf{Power}_{\mathsf{OReilly}}(x,y,B,A) \equiv\ & Ability(x,B,\mathsf{Claim}(x,y,A)) \vee Ability(x,B,\mathsf{Claim}(x,y,\overline{A})) \\ & \vee Ability(x,B,\neg\mathsf{Claim}(x,y,A)) \vee Ability(x,B,\neg\mathsf{Claim}(x,y,\overline{A}))\end{aligned}$$

[3] This position is neglected in the analytical literature but it is critically important in institutional-construction terms: it posits the denial to recognize another normative system which attempts to positively enact a certain power, see e.g. the Dutch Declaration of Independence, the Act of Abjuration (1581) towards Spain. [7]

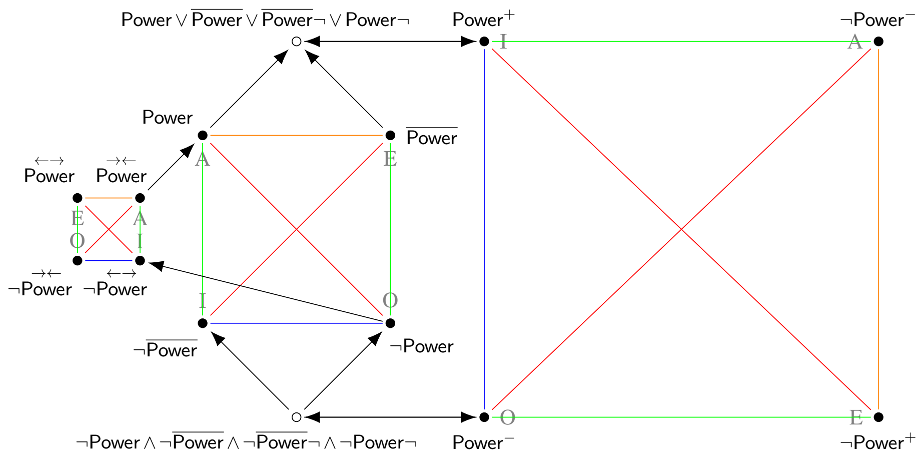

Figure 3. Map of potestative relations defined in terms of triggering action (force-centered square of opposition, the left one), in terms of outcome (middle square), in terms of change or affecting outcomes (change–centered square of opposition, the right one). For visual clarity, labels of vertices are simplified so as to consist only of a (possibly negated) predicate without its arguments. Occurrences of negation before a predicate are standard, whereas occurrences after a predicate denote action complementation; for instance, we denote the power to issue a prohibition, i.e., $\mathsf{Power}(p,q,\overline{A})$, as $\mathsf{Power}\neg$. Notice that the leftmost square is vertically mirrored and the rightmost square underwent a 90 degree clockwise rotation. Colours are as usual.

A *positive change-centered* power corresponds to the ability of affecting the target relation by any triggering action:

$$\mathsf{Power}^+(x,y,A) \equiv \exists\beta : \mathsf{Power}_{\mathsf{OReilly}}(x,y,\beta,A)$$

A *negative change-centered* power corresponds to the ability of the agent to perform an action without affecting the target relation:

$$\mathsf{Power}^-(x,y,A) \equiv \exists\beta : \neg\mathsf{Power}_{\mathsf{OReilly}}(x,y,\beta,A)$$

Labels for an Aristotelian square are, this time:

$$\mathsf{PR}^{\pm} = \{\mathsf{Power}^+(p,q,A), \mathsf{Power}^-(p,q,A), \neg\mathsf{Power}^-(p,q,A), \neg\mathsf{Power}^+(p,q,A)\}$$

Subalternity is here encoded by: $\neg\mathsf{Power}^-(x,y,A) \to \mathsf{Power}^+(x,y,A)$.

2.3.4. Relationships amongst powers

The previous formulas can be applied to discover different relationships between the distinct forms of powers. First, the convergence or divergence with due performance in force-centered powers map directly or dually to outcome-centered powers:

$$\overset{\rightarrow\leftarrow}{\mathsf{Power}}(x,y,A) \to \mathsf{Power}(x,y,A) \qquad \overset{\leftarrow\rightarrow}{\mathsf{Power}}(x,y,A) \to \mathsf{Power}(x,y,\overline{A})$$

Following the contrapositive, the absence of positive-outcome power to produce a duty (meaning that there is no triggering action to obtain this) maps *a fortiori* to the absence of a positive-force power for doing so:

$$\neg\mathsf{Power}(x,y,A) \rightarrow \neg\overset{\rightarrow\leftarrow}{\mathsf{Power}}(x,y,A) \qquad \neg\mathsf{Power}(x,y,\overline{A}) \rightarrow \neg\overset{\leftarrow\rightarrow}{\mathsf{Power}}(x,y,A)$$

Second, positive-change power holds if any outcome-center power holds:

$$\begin{aligned}\mathsf{Power}^{+}(x,y,A) \leftrightarrow\ & \mathsf{Power}(x,y,A) \vee \mathsf{Power}(x,y,\overline{A}) \\ & \vee \overline{\mathsf{Power}}(p,q,A) \vee \overline{\mathsf{Power}}(p,q,\overline{A})\end{aligned}$$

Assuming again that there is always some available action, we have, dually, that:

$$\begin{aligned}\mathsf{Power}^{-}(x,y,A) \leftrightarrow\ & \neg\mathsf{Power}(x,y,A) \wedge \neg\mathsf{Power}(x,y,\overline{A}) \\ & \wedge \neg\overline{\mathsf{Power}}(p,q,A) \wedge \neg\overline{\mathsf{Power}}(p,q,\overline{A})\end{aligned}$$

Introducing those relationships, we obtain the Aristotelian diagram in Fig. 3.

3. Diagrammatic theories

It is possible to define logical theories of different strength based on an Aristotelian diagram. These can be called *diagrammatic theories*. A diagrammatic theory over a diagram $\mathscr{D}$ encodes at least all logical relations among formulas used as labels in $\mathscr{D}$. A diagrammatic theory will be here defined in terms of the notion of an *inference tree*.

Definition 3 (Inference Tree)**.** *An* inference tree *T is an irreflexive and intransitive tree $(N, \rightsquigarrow)$ with a single* root *($N \neq \varnothing$ is a finite set of* nodes *and, for any $n,m \in N$, $n \rightsquigarrow m$ means that m is an* immediate successor *of n) where each $n \in N$ is associated with a finite set of formulas $\Gamma \neq \varnothing$ and has a* rank*. The rank of the root is $\mathbf{0}$; furthermore, if $rank(n) = \mathbf{i}$, n is associated with a set Θ, and $n \rightsquigarrow m$, then $rank(m) = \mathbf{i}+\mathbf{1}$ and m is associated with a set $\Sigma \supseteq \Theta$. Nodes with no successors are said to be* leaves *of T. A maximal $\rightsquigarrow$-chain of nodes $\sigma = (n_1,...,n_k)$ is a* branch *of T.*

Formulas in sets associated with nodes of a tree T can be uniformly substituted. Furthermore, an *equivalence relation* $Eq \subset \mathscr{L} \times \mathscr{L}$ can be established in order to replace, in any set Γ associated with a node n, a formula ϕ with a formula ψ, provided that $Eq(\phi, \psi)$.

Definition 4 (Set Immediate Inference)**.** *If Δ is a set of formulas (associated with a node) ranked with $\mathbf{i}$ and Δ' a set of formulas (associated with a node) ranked with $\mathbf{i}+\mathbf{1}$ in a branch σ of a tree T, then we say that Δ' can be immediately inferred from Δ within σ.*

Definition 5 (Diagrammatic Theory)**.** *A* diagrammatic theory *$\mathbb{DT}$ based on a diagram $\mathscr{D}$ is a set of inference trees satisfying the following properties, for each formulas ϕ and ψ that label some vertex of $\mathscr{D}$:*

- *if ψ is a* subalternant *of ϕ in $\mathscr{D}$, then some branch σ of a tree T in $\mathbb{DT}$ encodes an inference of the form $\Delta \cup \{\phi\} \rightsquigarrow \Delta \cup \{\phi, \psi\} \cup \Gamma$;*
- *if ψ and ϕ are* contraries *in $\mathscr{D}$, then some branch σ of a tree T in $\mathbb{DT}$ encodes an inference of the form $\Delta \cup \{\phi\} \rightsquigarrow \Delta \cup \{\phi, \neg\psi\} \cup \Gamma$;*
- *if ψ and ϕ are* sub-contraries *in $\mathscr{D}$, then some branch σ of a tree T in $\mathbb{DT}$ encodes an inference of the form $\Delta \cup \{\neg\phi\} \rightsquigarrow \Delta \cup \{\neg\phi, \psi\} \cup \Gamma$;*
- *if ψ and ϕ are* contradictories *in $\mathscr{D}$, then some branch σ of a tree T in $\mathbb{DT}$ encodes an inference of the form $\Delta \cup \{\phi\} \rightsquigarrow \Delta \cup \{\phi, \neg\psi\} \cup \Gamma$ and some branch σ' of a tree T' in $\mathbb{DT}$ encodes an inference of the form $\Delta' \cup \{\psi\} \rightsquigarrow \Delta' \cup \{\neg\phi, \psi\} \cup \Gamma'$.*

According to Definitions 4 and 5, the relation of immediate inference in a diagrammatic theory based on a diagram $\mathscr{D}$ encodes at least all Aristotelian relations between formulas labelling vertices of $\mathscr{D}$. However, the use of the equivalence relation Eq allows one to reduce the number to Aristotelian relations to be encoded, as we will clarify below. Here we just assume $Eq(\phi, \neg\neg\phi)$, for every $\phi \in \mathscr{L}$.

T' is a *sub-tree* of T iff the root of T' is a node $n \in T$ and the other nodes of T' are all those that (i) occur in branches of T to which n belongs, and (ii) have a higher rank than n in those branches. The cardinality of a set of formulas Γ will be denoted as $|\Gamma|$. The notion of inference within a branch is obtained by combining the transitive closure of the notion of immediate inference and the subset relation, as indicated below.

Definition 6 (Set Inference)**.** *A set of formulas* Γ *can be* inferred *from a set of formulas* Δ *within a branch* σ *of a tree* T *iff there is* $\Gamma' \supseteq \Gamma$ *s.t.:*

- *both* Δ *and* Γ' *belong to* σ*;*
- *the rank of* Γ' *in* σ *is not lower than the rank of* Δ *in* σ*.*

According to Definition 6, if $\Gamma \subseteq \Delta$, then Γ can always be inferred from Δ within a branch of a tree. In order to check whether Γ can be inferred from Δ in a branch σ of a tree T, one has to compare pairs of sets (checking whether they are identical, one is a subset of the other, etc.). Derivation is a particular kind of inference, as per the following definition.

Definition 7 (Set Derivability - Trees)**.** *A set of formulas* Γ *is derivable from a set of formulas* Δ *within a tree* T *iff there is a sub-tree* T' *of* T *whose root is a node* n *labelled by* Δ *and* Γ *can be inferred from* Δ *within all branches of* T'*.*

The derivability of Γ from Δ *within a diagrammatic theory* $\mathbb{DT}$ is defined in terms of a finite sequence of derivations within trees of $\mathbb{DT}$, as below.

Definition 8 (Set Derivability - Diagrammatic Theories)**.** *A set of formulas* Γ *can be derived from a set of formulas* Δ *within a diagrammatic theory* $\mathbb{DT}$ *iff there are trees* $T_1, ..., T_{n-1}$ *in* $\mathbb{DT}$ *and sets of formulas* $\Delta_1, ..., \Delta_n$ *s.t.:*

- $\Delta = \Delta_1$ *and* $\Gamma = \Delta_n$*;*
- *for* $1 \leq j < n$*,* Δ_{j+1} *can be derived from* Δ_j *within tree* T_j*.*

Example Below is an example of an inference tree that can be used in a diagrammatic theory built over Fig. 3. It captures inferences from a set of formulas Δ including the label of the E-corner in the change-centered (rightmost) square. Each line represents a node of the tree (which, in this, case has no branches) and starts with the node's rank:

$$\mathbf{0} : \Delta_0 = \Delta \cup \{\neg \mathsf{Power}^{+}(p,q,A)\} \rightsquigarrow$$
$$\mathbf{1} : \Delta_1 = \Delta_0 \cup \{\mathsf{Power}^{-}(p,q,A)\} \rightsquigarrow$$
$$\mathbf{2} : \Delta_2 = \Delta_1 \cup \{\neg \mathsf{Power}(p,q,A) \wedge \neg \overline{\mathsf{Power}}(p,q,A) \wedge \neg \overline{\mathsf{Power}}(p,q,\overline{A}) \wedge \neg \mathsf{Power}(p,q,\overline{A})\} \rightsquigarrow$$
$$\mathbf{3} : \Delta_3 = \Delta_2 \cup \{\neg \overline{\mathsf{Power}}(p,q,A), \neg \mathsf{Power}(p,q,A)\} \rightsquigarrow$$
$$\mathbf{4} : \Delta_4 = \Delta_3 \cup \{\neg \overleftrightarrow{Power}(p,q,A)\}$$

Any diagrammatic theory including this tree allows one (due to Definitions 4, 6 and 7) to derive any set $\Gamma \subseteq \Delta_i$, for $0 \leq i \leq 4$, from the starting set of formulas Δ_0.

4. Decidability and complexity

Given two *finite* sets $\Gamma, \Delta \subset \mathscr{L}$, we will say that the problem of checking whether Γ can be derived from Δ within a diagrammatic theory $\mathbb{DT}$ is the *derivability problem for finite sets* in $\mathbb{DT}$. We now illustrate that such a problem can be effectively computed. First, we need to define an auxiliary notion.

Definition 9 (Tree traversal)**.** *The* traversal *of a tree T with reference to a formula ϕ and a set Δ is a procedure which can be described as follows (we assume that Δ occupies the root of T):*

- *Following the order of ranks, for any set of formulas Γ with rank $\mathbf{i}$ in T, we compare ϕ with all formulas in Γ and keep track of whether ϕ occurs in Γ or not.*
- *The procedure terminates when either (*positive outcome*) there is a rank $\mathbf{j}$ s.t. all sets of formulas with rank $\mathbf{j}$ include ϕ or (*negative outcome*) all sets of formulas with all ranks available in T have been checked.*

Notice that, in case of a positive outcome of the traversal, due to Definitions 6 and 7, $\Delta \cup \{\phi\}$ is derivable from Δ within T. If the number of nodes in T is l, and $max\{|\Theta \setminus \Sigma| : rank(\Theta) = \mathbf{1} + rank(\Sigma)\} = k$ (hereafter, *the maximum successor difference*), then a traversal of T with reference to ϕ and Δ requires up to $k * l$ moves.

Definition 10 (Theory traversal)**.** *The* traversal of a diagrammatic theory $\mathbb{DT}$ *with reference to a formula ϕ and a set of formulas Δ is the traversal of all trees T in $\mathbb{DT}$ with reference to ϕ and Δ. The outcome is* positive *iff it is positive for some T in $\mathbb{DT}$.*

If the number of trees in $\mathbb{DT}$ is h, the maximum number of nodes in a tree of $\mathbb{DT}$ is l, and the maximum successor difference in a tree of $\mathbb{DT}$ is k, then a traversal of $\mathbb{DT}$ with reference to ϕ and Δ requires up to $(k * l) * h$ moves.

Theorem 1 (Decidability)**.** *The derivability problem for finite sets within a diagrammatic theory is decidable.*

Proof. Consider two finite sets of formulas Δ and Γ, such that $max(|\Delta|, |\Gamma|) = n$. Let $\mathbb{DT}$ be the diagrammatic theory at issue, consisting of h inference trees. To see whether Γ is derivable from Δ, we first need to compare the sets Γ and Δ, an operation with time complexity $O(n)$. There are four relevant Cases: (1) $\Gamma = \Delta$; (2) $\Gamma \subset \Delta$; (3) $\Gamma \supset \Delta$; (4) $\Gamma \not\subset \Delta$, $\Gamma \not\supset \Delta$ and $\Gamma \neq \Delta$. In Cases 1 and 2, by Definition 8, we can immediately conclude that Γ is derivable from Δ within $\mathbb{DT}$. In Cases 3 and 4 one considers the set $\Gamma \setminus \Delta$. Let $|\Gamma \setminus \Delta| = m$. We know that the elements of $\Gamma \setminus \Delta$ can be enumerated ($m < n$ by construction). We take the first formula ϕ_1 in $\Gamma \setminus \Delta$ and perform a *traversal* of $\mathbb{DT}$ with reference to ϕ_1 and Δ. If the traversal produces a negative outcome, then the whole procedure terminates and Γ is not derivable from Δ. Otherwise, there is a tree T s.t., due to Definitions 6, 7 and 9, $\Delta \cup \{\phi\}$ is derivable from Δ within T. We take $\Delta_1 = \Delta \cup \{\phi_1\}$ and then perform a *traversal* of $\mathbb{DT}$ with reference to ϕ_2 and Δ_1. The procedure is reiterated: at each step a traversal of $\mathbb{DT}$ is performed with reference to a new formula $\phi_j \in \Gamma \setminus \Delta$ and the set of formulas Δ_{j-1} obtained at the previous step. In accordance with Definition 8, Γ is derivable from Δ iff each traversal of $\mathbb{DT}$ performed ends with a positive outcome. $\square$

Theorem 2 (Complexity)**.** *The algorithm to solve the derivability problem for finite sets in a diagrammatic theory takes polynomial time.*

Proof. In all Cases (1-4) mentioned in the proof of Theorem 1 the procedure terminates in at most $((k*l)*h)*m$ moves. □

5. Final remarks

Recent years have witnessed a renewed interest in diagrams and logical geometry: our work provides a further contribution towards their application in the analysis of normative problems. We introduced a syntactic method of reasoning over Aristotelian diagrams that encode relations between normative positions. The method can be automated and allows one to derive a finite set of normative positions from another in polynomial time. In our opinion, reasoning methods associated with diagrams are very promising and, due to the cognitive efficacy of visual aids, they are potentially accessible to a broader audience. Future investigations concern: (i) a theoretical and empirical comparison with other methods for (normative) automated reasoning, (ii) an integration of temporal/causal components within the definition of normative positions, and (iii) the consolidation of a structured taxonomy of Aristotelian diagrams for normative reasoning.

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