Solutions to Games, Transitions and Efficiency

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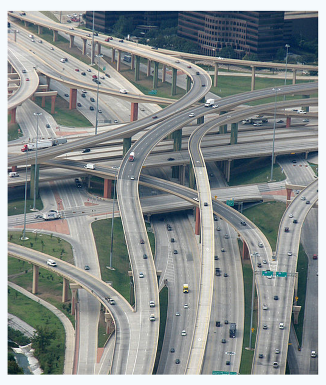
We model lack of coordination on the same solution and analyse the efficiency repercussions of this lack of coordination.

Example: Lack of Coordination

- Some people are active, while some are not
- Various marriage concepts at the same time
- Different equilibrium flows in a road network



Introduction

Nash equilibrium suffers from
- Strong belief assumptions
- Non-simultaneous change
- Lack of coordination

We model these issues as a transition and bound its efficiency.

Model

1. A game $G = (N, S = S_1 \times S_2 \times \ldots \times S_n, (u_i)_{i=1,\ldots,n})$
2. A solution concept (e.g., NE) defines a solution set $D \subseteq S$

Definition 1 *Given a game G and $D \subseteq S$, define a* transition *as any profile $s = (s_1, \ldots, s_n) \in S$ such that for each $i \in N$, there exists a solution $d(s, i) = (d_1, \ldots, d_n) \in D$, such that $s_i = d_i$. Denote the* transition set *(set of all the transitions) as $T(D) \subseteq S$.*

Definition 2 *An m-transition allows for mixing at most m solutions. Denote the m-transition set by $T(D, m)$.*

By definition, $D \subseteq T(D, m) \subseteq T(D, n) = T(D)$ and $T(T(D)) = T(D)$.

For game $G = (N, S, (u_i)_{i=1,\ldots,n})$ and solution set $D \subseteq S$,
- $\mathrm{SW}(s) \triangleq \sum_{i \in N} u_i(s)$
- $\mathrm{PoA} \triangleq \frac{\min_{s \in D} \mathrm{SW}(s)}{\max_{s \in S} \mathrm{SW}(s)}$ and $\mathrm{PoS} \triangleq \frac{\max_{s \in D} \mathrm{SW}(s)}{\max_{s \in S} \mathrm{SW}(s)}$

We define
- $\mathrm{PoTA} \triangleq \frac{\min_{s \in T(D)} \mathrm{SW}(s)}{\max_{s \in S} \mathrm{SW}(s)}$ and $\mathrm{PoTS} \triangleq \frac{\max_{s \in T(D)} \mathrm{SW}(s)}{\max_{s \in S} \mathrm{SW}(s)}$
- $m-\mathrm{PoTA} \triangleq \frac{\min_{s \in T(D,m)} \mathrm{SW}(s)}{\max_{s \in S} \mathrm{SW}(s)}$ and $m-\mathrm{PoTS} \triangleq \frac{\max_{s \in T(D,m)} \mathrm{SW}(s)}{\max_{s \in S} \mathrm{SW}(s)}$

General Efficiency Bounds

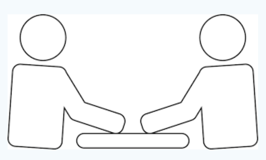
For any solution concept, there always holds $\mathrm{PoTA} \leq \mathrm{PoA}, \mathrm{PoTS} \geq \mathrm{PoS}$. In the opposite direction, bounds on personal utilities in a transition imply bounds on the PoTA, PoTS, $m-$PoTA, and $m-$PoTS.

In a constant-sum game, $\mathrm{PoA} = \mathrm{PoS} = \mathrm{PoTS} = \mathrm{PoTA} = 1$.

Theorem 1 *In a congestion game with subadditive cost functions, $m-\mathrm{PoTA} \leq m\,\mathrm{PoA}$, and this is tight.*

A game where the players have equal number of strategies can be decomposed to a zero-sum game and a potential game (Candogan et al. 2011). We connect the efficiency of the game and its potential part.

Proposition 1 *In a 2-player game, condition $u_1(x, y) \leq u_1(x', y)$ and $u_2(x, y) \leq u_2(x, y') \Rightarrow \mathrm{SW}(x, y) \leq \mathrm{SW}(x', y)$ or $\mathrm{SW}(x, y) \leq \mathrm{SW}(x, y')$ implies $\mathrm{PoTS} = \mathrm{PoS}$.*



We define *extensive smoothness*, which allows bounding the PoTA.

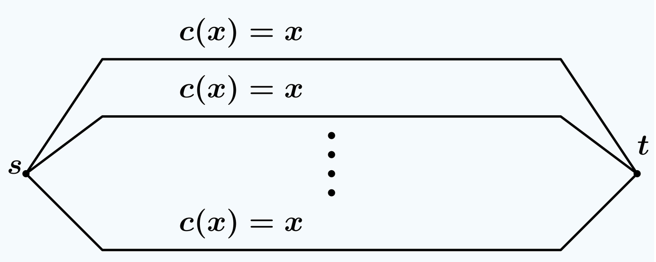
Proposition 2 *For an identical utility game, the $\mathrm{PoS} = \mathrm{PoTS} = 1$, but the price of anarchy can be arbitrarily low, and the PoTA can be arbitrarily low relatively to the PoA.*
If we also have that the best response strategies of any player i to the strategies s_{-i} of the others do not depend on those s_{-i}, then $\mathrm{PoTA} = \mathrm{PoA} = \mathrm{PoS} = \mathrm{PoTS} = 1$.

	1 :	2 :
I :	(ϵ, ϵ)	$(0, 0)$
II :	$(0, 0)$	(a, a)

Routing Games

1. Route a commodity of size r_i from s_i to t_i through paths $\mathcal{P}_i$
2. An *equilibrium flow* is a feasible flow f where every used path is optimum with respect to cost c_e
3. Define the PoA as $\frac{\text{the cost of the equilibrium flow}}{\text{the optimum cost}}$
4. Define a *transition* as a feasible flow such that $f_P > 0 \Rightarrow$ there exists an equilibrium flow f' with $f'_P > 0$
5. Define the PoTA (PoTS) as $\frac{\text{the cost of a most costly (cheapest) transition}}{\text{the optimum cost}}$

Example 1 • *1 commodity*
- *1 eq. flow and continuum transitions*
- $\mathrm{PoA} = \mathrm{PoS} = \mathrm{PoTS} = 1$, *but* $\mathrm{PoTA} = n$



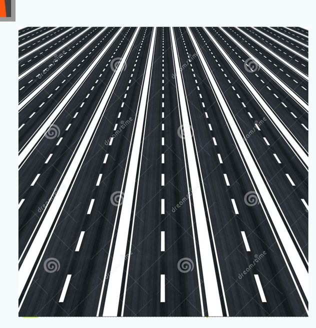
Theorem 2 *For cost functions $\mathcal{C}$ and a commodity i, define*

$$S_i(\mathcal{C}) \triangleq \frac{\max\{|P| : P \in \mathcal{P}_i\} \sup_{c \in C}(c(r_i + \sum_{j \in \{1,\ldots,k\} \setminus \{i\}} r_j))}{\min\{|P| : P \in \mathcal{P}_i\} \inf_{c \in C} c(r_i / |\mathcal{P}_i|)}.$$

Then, $\mathrm{PoTA} \leq \mathrm{PoA} \cdot \max_{i=1,\ldots,k} S_i(\mathcal{C})$, and this bound is tight.

Results and Conclusions

1. Most efficiency bounds are not promising $\Rightarrow$ coordinate
2. The bounds are optimistic for
   - potential game and low transition degree
   - identical utility game with independent best responses
   - routing with linear and close cost functions, non-intersecting commodities, similar path lengths, and few paths per commodity



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