Solutions to Games, Transitions and Efficiency

Gleb Polevoy Paola Grosso Cees de Laat

SNE, The University of Amsterdam, The Netherlands

The Phenomena

A solution concept, such as Nash equilibrium

- Strong belief assumptions
- Non simultaneous change (democracy, marriage, traffic)
- Lack of coordination

No theoretical modelling of using various solutions simultaneously

⇒ We

1. formally model a transition
2. bound efficiency

Model

Given

1. A game $G = (N, S = S_1 \times S_2 \times \ldots \times S_n, (u_i)_{i=1,\ldots,n})$
2. A solution concept (e.g., NE) defines a solution set $D \subseteq S$

To model movement or lack of coordination,

Definition

Given $D \subseteq S$, define
a transition as any profile $s = (s_1, \ldots, s_n) \in S$ such that for each $i \in N$, there exists a solution $d(s, i) = (d_1, \ldots, d_n) \in D$, such that $s_i = d_i$. Denote the set of all the transitions to be $T(D) \subseteq S$, the transition set.

By definition, $D \subseteq T(D)$ and $T(T(D)) = T(D)$

Model - Efficiency

Classically,

- $\mathrm{SW}(s) \triangleq \sum_{i \in N} u_i(s)$
- $\mathrm{PoA} \triangleq \frac{\min_{s \in D} \mathrm{SW}(s)}{\max_{s \in S} \mathrm{SW}(s)}$ and $\mathrm{PoS} \triangleq \frac{\max_{s \in D} \mathrm{SW}(s)}{\max_{s \in S} \mathrm{SW}(s)}$

Given

1. $G = (N, S, (u_i)_{i=1,\ldots,n})$
2. $D \subseteq S$

We define

- $\mathrm{PoTA} \triangleq \frac{\min_{s \in T(D)} \mathrm{SW}(s)}{\max_{s \in S} \mathrm{SW}(s)}$ and $\mathrm{PoTS} \triangleq \frac{\max_{s \in T(D)} \mathrm{SW}(s)}{\max_{s \in S} \mathrm{SW}(s)}$

General Bounds

Always holds

$$\text{PoTA} \leq \text{PoA}, \text{PoTS} \geq \text{PoS},$$

but not the other direction, generally speaking

General Bounds - Individual Utilities

Definition

Player i's utility over profile set $A \subseteq S$ is α-lower (-upper) dependent on coordination if

$$\min_{s \in T(A)} u_i(s) \geq \min_{t \in A} u_i(t)/\alpha \qquad (\max_{s \in T(A)} u_i(s) \leq \alpha \cdot \max_{t \in A} u_i(t)).$$

Definition

The utility of agent i is β varied over $A \subseteq S$ if for all profiles s, t in A,

$$\mathrm{SW}(s) \geq \mathrm{SW}(t) \Rightarrow u_i(s) \geq u_i(t)/\beta.$$

For example, the utility of a game with identical payoff functions is 1-upper dependent on coordination and 1 varied over any set

General Bounds - Individual Utilities

Proposition

Consider a game $G = (N, S, (u_i)_{i=1,\dots,n})$ with a solution set $D \subseteq S$, such that over D, the utility of every player i is β varied and α-lower dependent on coordination, then

$$\mathrm{PoTA} \geq \mathrm{PoA} / (\alpha\beta). \tag{1}$$

If for every player i, its utility over D is β varied and α-upper dependent on coordination, then

$$\mathrm{PoTS} \leq \alpha\beta\, \mathrm{PoS}\,. \tag{2}$$

For example, an identical utility game has PoTS = PoS

Nash Equilibria Bounds - Two Players

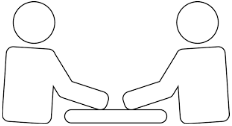
Now, concentrate on NE and $T(NE)$

Proposition

In a two-player game, if for every $x, x' \in S_1$ and every $y, y' \in S_2$ there holds the implication

$$u_1(x, y) \leq u_1(x', y) \text{ and } u_2(x, y) \leq u_2(x, y') \Rightarrow \mathrm{SW}(x, y) \leq \mathrm{SW}(x', y) \text{ or } \mathrm{SW}(x, y) \leq \mathrm{SW}(x, y'), \tag{3}$$

then we have $\mathrm{PoTS} = \mathrm{PoS}$.



Nash Equilibria Bounds - Extensive Smoothness

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A game is $\alpha, \beta, \lambda, \mu$*-extensively smooth if the following holds:*

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1. $\forall s^* \in \arg\max \{SW(s) : s \in S\}, \forall t \in T(NE) : \sum_{i=1}^{n} u_i(s_i^*, t_{-i}) \geq \lambda SW(s^*) - \mu SW(t)$.

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2. $\forall i \in N, \forall s \in T(NE), \forall d \in NE$ *such that* $s_i = d_i : u_i(s) \geq \alpha u_i(d)$.

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3. $\forall s^* \in \arg\max \{SW(s) : s \in S\}, \forall t, v \in T(NE) : u_i(s_i^*, t_{-i}) \geq \beta u_i(s_i^*, v_{-i})$.

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Proposition

Any $\alpha, \beta, \lambda, \mu$-extensively smooth game has PoTA $\geq \frac{\alpha\beta\lambda}{1+\alpha\beta\mu}$.

Nash Equilibria Bounds - Identical Utility Games

Proposition

For an identical utility game, the PoS $=$ PoTS $= 1$*, but the price of anarchy can be arbitrarily low.*
If we also have that the best response strategies of any player i to the strategies s_{-i} of the others do not depend on those s_{-i}, then PoTA $=$ PoA $=$ PoS $=$ PoTS $= 1$.

	1 :	2 :
I :	(ϵ, ϵ)	$(0, 0)$
II :	$(0, 0)$	(a, a)

Non-Atomic Routing Game Bound - Definitions

Definition

1. *Source and sink pairs* $(s_1, t_1), \ldots, (s_k, t_k)$
2. *Each commodity is of size* r_i *to be routed through paths in* $\mathcal{P}_i$
3. *A flow vector* $f \in \mathbb{R}_+^{|\mathcal{P}|}$ *is feasible if* $\sum_{P \in \mathcal{P}_i} f_P = r_i$
4. *Each edge has a non-decreasing cost function* $c_e : \mathbb{R}_+ \to \mathbb{R}_+$
5. *Define* $c_P(f) \triangleq \sum_{e \in P} c_e(f_e)$
6. *Define an equilibrium flow as a feasible flow* f *such that for every commodity* $i = 1, \ldots, k$, *for every path* $P \in \mathcal{P}_i$ *such that* $f_P > 0$ *and for every path* $P' \in \mathcal{P}_i$ *we have* $c_P(f) \leq c_{P'}(f)$

Non-Atomic Routing Game Bound - Definitions Cont.

Definition

1. *Define* $c_P(f) \triangleq \sum_{e \in P} c_e(f_e)$
2. *An equilibrium flow*

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2. *An equilibrium flow*
3. $C(f) \triangleq \sum_{P \in \mathcal{P}} c_P(f) \cdot f_P$
4. *Define the* PoA *as* $\frac{\textit{the cost of the equilibrium flow}}{\textit{the optimum cost}}$
5. *Define a* ***transition*** *as a feasible flow such that* $f_P > 0 \Rightarrow$ *there exists an equilibrium flow* f' *with* $f'_P > 0$
6. *Define the* PoTA *(*PoTS*) as* $\frac{\textit{the cost of a most costly (cheapest) transition}}{\textit{the optimum cost}}$

Non-Atomic Routing Game Bound - Example

Example

- The only commodity with $r = 1$
- One equilibrium and a continuum of transitions
- $\text{PoA} = \text{PoS} = \text{PoTS} = 1$
- However, $\text{PoTA} = n$

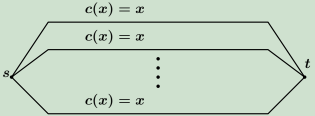

Figure: Having n parallel edges with $c_e(x) = x$ each.

Non-Atomic Routing Game Bound

Theorem

Given a set of cost functions $\mathcal{C}$, a routing game and a commodity i, define

$$S_i(\mathcal{C}) \triangleq \frac{\max\{|P| : P \in \mathcal{P}_i\} \sup_{c \in C} (c(r_i + \sum_{j \in \{1,\ldots,k\} \setminus \{i\}} r_j))}{\min\{|P| : P \in \mathcal{P}_i\} \inf_{c \in C} c(r_i / |\mathcal{P}_i|)}. \quad (4)$$

Then, $\mathrm{PoTA} \leq \mathrm{PoA} \cdot \max_{i=1,\ldots,k} S_i(\mathcal{C})$, *and this bound is tight.*

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Then, $\text{PoTA} \leq \text{PoA} \cdot \max_{i=1,\ldots,k} S_i(\mathcal{C})$*, and this bound is tight.*
In particular, if $c_e(x) = a_e \cdot x$, such that $a_{\min} \leq a_e \leq a_{\max}$ and also the paths of different commodities never intersect, then

$$S_i(\mathcal{C}) = \frac{\max\{|P| : P \in \mathcal{P}_i\} a_{\max}}{\min\{|P| : P \in \mathcal{P}_i\} a_{\min}} |\mathcal{P}_i|. \quad (5)$$

Conclusions

1. Modelling lack of coordination
2. General efficiency bounds are appalling $\Rightarrow$ coordinate
3. Most NE bounds are not promising $\Rightarrow$ coordinate
4. The bounds are optimistic for
   - identical utility game with independent best responses
   - routing games with linear costs, non-intersecting commodities, similar path lengths per commodity, close cost functions, and few paths per commodity

Future Work

- Limited transitions
- Repeated game
- Combining solutions from different solution concepts

Thank You!